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Study on cavitated bifurcation problems for spheres composed of hyper-elastic materials

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Abstract. In this paper, spherical cavitated bifurcation problems are examined for incompressible hyper-elastic materials and compressible hyper-elastic materials, respectively. For incompressible hyper-elastic materials, a cavitated bifurcation equation that describes cavity formation and growth for a solid sphere, composed of a class of transversely isotropic incompressible hyper-elastic materials, is obtained. Some qualitative properties of the solutions of the cavitated bifurcation equation are discussed in the different regions of the plane partitioned by material parameters indicating the degree of radial anisotropy in detail. It is shown that the cavitated bifurcation equation is equivalent, by use of singularity theory, to a class of normal forms with single-sided constraint conditions at the critical point. Stability and catastrophe of the solutions of the cavitated bifurcation equation are discussed by using the minimal potential-energy principle. For compressible hyper-elastic materials, a group of parameter-type solutions for the cavitated deformation for a solid sphere, composed of a class of isotropic compressible hyper-elastic materials, is obtained. Stability of the solutions is also discussed.

Key words: cavitated bifurcation, critical dead load and stretch, hyper-elastic material, secondary bifurcation point, singularity theory

1. Introduction

In 1958, Gent and Lindley [1] discovered rubber cavitation in the laboratory for the first time. Many similar phenomena have been observed since then (see [2–4]). Void formation, growth and run-through of the adjacent cavity in solids have been of concern for a long time because of the important role that such phenomena play in failure and fracture mechanisms.

In 1982, Ball [5] formulated the phenomenon of void nucleation and growth as a class of bifurcation problems in the context of nonlinear elasticity for the first time, which established the basic theory for such problems. Thereafter, many significant results have been obtained.

For incompressible hyper-elastic materials, an explicit formula to determine the critical dead load for the cavity formation was given by Ball [5]. Cavitated bifurcation problems for spheres composed of isotropic hyper-elastic materials of power-law type, the anisotropic neo-Hookean material, the isotropic Valanis-Landel material and the transversely isotropic Ogden material have been investigated by Chou-Wang and Horgan [6], Polignone and Horgan [7], Ren and Cheng [8, 9], respectively. But it has been shown that cavitation can appear only in the case of finite strains (see [10]). Further references may be found in [11, 12]. See the review article by Polignone and Horgan [13] for a comprehensive review of results up to 1995 for both incompressible and compressible materials.

However, for compressible hyper-elastic materials, the study of cavitated bifurcation is more difficult due to the inherent nonlinearity of the problems. Ball [5] gave radially symmetric solutions of n-dimensional boundary-value problems for the displacement and analyzed the existence and stability of bifurcation solutions for compressible hyper-elastic materials. In 1986, Horgan and Abeyaratne [14] carried out the analyses on growth of pre-existing micro-voids for the Blatz-Ko material. Thereafter, Biwa *et al.* [15] studied the effect of constitutive parameters on the formation of a spherical void in a class of compressible nonlinear elastic materials. See the review article [13] for details. Exact solutions for cavitated bifurcation for the generalized Varga material, the generalized Carroll material, the modified Blatz-Ko material and a class of compressible hyper-elastic materials were presented by Horgan [16], Murphy and Biwa [17], Shang and Cheng [18, 19]. Recently, Ren and Cheng [20] studied void nucleation and growth for composite compressible hyper-elastic materials. Kakavas [21] studied the influence of cavitation on the stress-strain fields of the compressible Blatz-Ko material in the case of finite deformations. Further references are [22–24].

As is well known, the constitutive law of a hyper-elastic material may be described completely by its strain-energy function. In 1955, Ericksen [25] proved an important result: the only deformations that are controllable for compressible isotropic materials are homogeneous deformation fields. Later, Rivlin [26], Varga [27] and Ogden [28] proposed several important models of strain-energy functions. In 1988, Carroll [29] introduced three classes of compressible isotropic hyper-elastic materials, which are expressed as functions of the principal invariants of the strain tensor, and presented the solutions for spheres composed of these materials in the case of finite strains. Thereafter, Murphy and Biwa [17] extended the strain-energy functions and obtained a class of generalized Carroll materials. Hill and Arrigo [30] proposed a modified Varga strain-energy function involving the reciprocals of the principal stretches, in 1995. They also derived new families of exact solutions, for plane and axially symmetric deformations by using transformation and reduced-equation methods in [30–32]. In 2001, Shang and Cheng [19] presented a linear approximated strain-energy function that was expressed as a function of another set of invariants of the strain tensor.

The purpose of this paper is to examine in detail spherical cavitated bifurcation problems for transversely isotropic incompressible hyper-elastic materials a compressible hyper-elastic materials. In Section 2, we formulate mathematical models for radially symmetric deformation problems for a solid sphere composed of a class of transversely isotropic incompressible hyper-elastic materials under a prescribed uniform surface dead load, and for a solid sphere composed of a class of compressible hyper-elastic materials under a prescribed surface displacement, respectively. In Section 3, for the transversely isotropic incompressible hyper-elastic sphere, a cavitated bifurcation equation, which describes cavity formation and growth in the interior of the sphere, is obtained. Some qualitative properties of the solutions of the cavitated bifurcation equation are discussed in detail in the different regions partitioned by material parameters indicating the degree of radial anisotropy. It is shown, by use of singularity theory, that the dimensionless cavitated bifurcation equation is equivalent to the normal forms with single-sided constraint conditions at the critical point. Stability and catastrophe of the solutions of the cavitated bifurcation equation are discussed by using the minimal potential-energy principle. At the end of this section, we point out that the phenomena of catastrophe and concentration of stresses occurring subsequent to cavitation coincide with the physical behavior for hyper-elastic materials. In Section 4, a group of parameter-type solutions for the cavitated deformation and the expression of critical stretch are obtained for the compressible hyper-elastic sphere under a prescribed surface displacement. In contrast to the critical radial stretch found by Shang and Cheng [19] for the occurrence of a cavity in the interior of the hyper-elastic sphere, the critical radial stretch associated with the strain-energy function given in this paper is smaller than that in [19] for the same Poisson ratio. Stability of the solutions is discussed as the prescribed stretch exceeds its critical value. With the appearance of a cavity, an interesting feature of the radial deformation near the deformed cavity wall is the transition from extension into compression.

2. Formulation

2.1. Basic governing equations

Consider the radially symmetric deformation of a solid here with radius A. Assume that the sphere is composed of a homogeneous hyper-elastic material. It is subjected to a prescribed uniform dead load $p_0 > 0$ or a prescribed uniform radial stretch λ, $(\lambda > 1)$ on its surface. In the spherical coordinate system, the occupied region of the undeformed solid sphere is denoted by

$$D_0 = \{(R, \Theta, \Phi)|0 \leq R < A, \quad 0 < \Theta \leq 2\pi, \quad 0 \leq \Phi \leq \pi\}.$$

The deformation is assumed to be radially symmetric; thus the deformed configuration is denoted by

$$D = \{(r, \theta, \phi)|r = r(R), \theta = \Theta, \phi = \Phi, 0 < R \leq A, r(0+) \geq 0\} \tag{1}$$

where $r = r(R) \geq 0$ is an undetermined function. If $r(0+) = 0$, the sphere remains solid, while if $r(0+) > 0$, a spherical cavity with radius $r(0+)$ forms at the center of the sphere. In this case, the cavity surface is assumed to be traction-free. From the radially symmetric deformation (1), the principal values of the strain tensor are given by

$$\lambda_1 = \dot{r}(R), \lambda_2 = \lambda_3 = r(R)/R, \tag{2}$$

where $\dot{r}$ denotes the derivative with respect to R. Assuming that $\lambda_1\lambda_2\lambda_3 > 0$ on $0 < R \leq A$, from (1), we have

$$\lambda_1 = \dot{r}(R) > 0, \quad 0 < R \leq A. \tag{3}$$

Thus, from $r(0+) \geq 0$, we deduce that $r(R) > 0$ on $0 < R \leq A$.

2.1.1. *Incompressible hyper-elastic sphere*

The incompressibility condition requires that $\lambda_1\lambda_2\lambda_3 = 1$, with (2), so we have $\dot{r}(R) = R^2/r^2(R)$ and

$$r(R) = (R^3 + c^3)^{1/3}. \tag{4}$$

Here, the Cauchy stresses in terms of the strain-energy function $W(\lambda_1, \lambda_2, \lambda_3)$ are given by

$$\tau_{rr}(R) = \lambda_1 \frac{\partial W}{\partial \lambda_1} - p(R), \quad \tau_{\theta\theta}(R) = \tau_{\phi\phi}(R) = \lambda_2 \frac{\partial W}{\partial \lambda_2} - p(R), \tag{5}$$

where $p(R)$ is an unknown hydrostatic pressure associated with the incompressibility condition. At the center of the sphere, we have the condition

$$r(0+)\tau_{rr}(0+) = 0. \tag{6}$$

Equation (6) means that, if no cavity occurs, $r(0+) = 0$, if a cavity forms at the center of the sphere, $r(0+) > 0$, hence, we have $\tau_{rr}(0+) = 0$.

Since the surface of the sphere is subjected to a prescribed uniform radial tensile dead load $p_0 > 0$, the surface condition requires that

$$\tau_{rr}(A) = p_0 \left[\frac{A}{r(A)}\right]^2. \tag{7}$$

2.1.2. *Compressible hyper-elastic sphere*

The Cauchy stresses in terms of the strain- energy function $W(\lambda_1, \lambda_2, \lambda_3)$ are given by

$$\tau_{rr}(R) = \frac{1}{\lambda_2\lambda_3}\frac{\partial W}{\partial \lambda_1}, \tau_{\theta\theta}(R) = \tau_{\phi\phi}(R) = \frac{1}{\lambda_1\lambda_2}\frac{\partial W}{\partial \lambda_3} = \frac{1}{\lambda_1\lambda_3}\frac{\partial W}{\partial \lambda_2}. \tag{8}$$

In this case, the condition at the center of the sphere is the same as (6). Since the surface of sphere is now subjected to a prescribed uniform radial stretch λ, the surface condition is given by

$$r(A) = \lambda A, \qquad (\lambda \geq 1). \tag{9}$$

The equilibrium equation for spherically symmetric deformation, in the absence of body forces, becomes

$$\dot{\tau}_{rr}(R) + 2\frac{\dot{r}(R)}{r(R)}(\tau_{rr}(R) - \tau_{\theta\theta}(R)) = 0, \tag{10}$$

where τ_{rr} and $\tau_{\theta\theta}$ are given by (5) for the incompressible material, and by (8) for the compressible material.

2.2. Strain-energy functions

As is well known, the response of an isotropic hyper-elastic material is described completely by its strain-energy function

$$W = W(\lambda_1, \lambda_2, \lambda_3), \tag{11}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal values of the strain tensor and $W(\lambda_1, \lambda_2, \lambda_3)$ are symmetric functions of λ_1, λ_2 and λ_3. The strain-energy function can also be written as

$$W = \widetilde{W}(j_1, j_2, j_3), \tag{12}$$

where j_1, j_2 and j_3 are the invariants of the stretch tensor and they are given by

$$j_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad j_2 = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}, \quad j_3 = \lambda_1\lambda_2\lambda_3. \tag{13}$$

Since $\widetilde{W}(j_1, j_2, j_3)$ is a linear function with respect to j_1, j_2 and j_3, the strain-energy function may be denoted by

$$W = \widetilde{W}(j_1, j_2, j_3) = c_1(j_1 - 3) + c_2(j_2 - 3) + (c_2 - c_1)(j_3 - 1). \tag{14}$$

To guarantee that the isotropic hyper-elastic materials represented by the strain-energy function (14) linearize properly to the classic linear theory (*cf.* Ogden [33, pp. 349]), it is easy to obtain $c_1 = \mu\dfrac{1-3\nu}{1-2\nu}$, $c_2 = \mu\dfrac{1-\nu}{1-2\nu}$, where μ is the shear modulus and ν is the Poisson ratio in the state of infinitesimal deformations. In [19], Shang and Cheng studied the cavitated bifurcation problem for this class of materials. When the material is incompressible, *i.e.*, $j_3 = \lambda_1\lambda_2\lambda_3 = 1$, the strain-energy function may be given as

$$W = \widetilde{W}(j_1, j_2, 1) = \mu\left[a(j_1 - 3) + b(j_2 - 3)\right], \tag{15}$$

where μa and μb are two material constants such that $a + b = 2$. Hill and Arrigo [30] called (15) the modified Varga strain-energy function, and they also derived new families of exact solutions for plane and axially symmetric deformations by using the transformation and reduced equation methods [30–32].

In this paper, we use these, with λ_1, λ_2 and λ_3 given by (2), to discuss the spherical symmetric deformation problems for solid spheres composed of two kinds of hyper-elastic materials, which are incompressible and compressible, respectively.

For incompressible materials, we assume that the strain-energy function is given by

$$W(\lambda_1, \lambda_2, \lambda_3) = \mu\left[a(\lambda_1+\lambda_2+\lambda_3-3)+b\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}-3\right)+\alpha(\lambda_1-1)^2+\beta(\lambda_1-1)^3\right], \tag{16}$$

where $\alpha, \beta \geq 0$ are the dimensionless material parameters, and they may be regarded as measures of the degree of anisotropy of the material about the radial direction. In particular, when $\alpha = \beta = 0$, the corresponding hyper-elastic material is isotropic, so (16) reduces to (15). If at least one of α, β is nonzero, the hyper-elastic material is called transversely isotropic about the radial direction. Polignone and Horgan [7] presented perhaps the first paper idealing with a cavitation-formation problem for this kind of material. In their work, the strain-energy functions of anisotropic materials were discussed in detail. In the present paper, we are mainly concerned with the effect of the material parameters, *i.e.*, α and β, on cavity formation and growth for spheres composed of the transversely isotropic hyper-elastic materials (16).

For compressible materials, assume that the strain-energy function has the form

$$W = \widetilde{W}(j_1, j_2, j_3) = c_1(j_1 - 3) + c_2(\ln j_2 - \ln 3) + c_3(j_3 - 1). \tag{17}$$

It is easy to show that the strain-energy function (17) satisfies certain constitutive inequalities imposed to ensure that the physical behavior of the material is realistic. Further, we have (*cf.* Ogden [33, pp. 349])

$$c_1 = \mu\frac{2(2-7\nu)}{5(1-2\nu)}, \quad c_2 = \mu\frac{18(1-\nu)}{5(1-2\nu)}, \quad c_3 = \mu\frac{2(1+4\nu)}{5(1-2\nu)}. \tag{18}$$

Obviously, the strain-energy function (17) has a unique local minimum (3,3,1), corresponding to the natural state of the material, and the minimum is always nonnegative if c_1, c_2, c_3 are positive, so we must assume that $\mu > 0$, $0 < \nu < 2/7$.

Thus, the mathematical model of the spherical symmetric deformation for a transversely isotropic incompressible hyper-elastic solid sphere, subjected to a uniform tensile dead load $p_0 > 0$ on its surface, consists of the Equation (4), (5), (10), (16), and the boundary conditions (6), (7). The mathematical model for a compressible hyper-elastic solid sphere, subjected to a prescribed uniform radial stretch $\lambda > 1$ on its surface, is formed by the Equations (2), (8), (10), (17) and the boundary conditions (6), (9).

3. Cavitated bifurcation for incompressible hyper-elastic sphere

First, on substituting (4) in (2), this yields $\lambda_1 = \left(1+\frac{c^3}{R^3}\right)^{-2/3}$, $\lambda_2 = \lambda_3 = \left(1+\frac{c^3}{R^3}\right)^{1/3}$. For convenience of calculation, we introduce the notation $\omega = \omega(R,c) = \lambda_2 = \lambda_3$, $\omega^{-2}(R,c) = \lambda_1$, and thus the strain-energy function (16) and the corresponding Cauchy stresses can be written as

$$\begin{aligned}\Sigma(\omega) &= W\left(\omega^{-2},\omega,\omega\right) \\ &= \mu\left[a\left(\omega^{-2}+2\omega-3\right)+b\left(2\omega^{-1}+\omega^2-3\right)+\alpha\left(\omega^{-2}-1\right)^2+\beta\left(\omega^{-2}-1\right)^3\right],\end{aligned} \tag{19}$$

$$\tau_{rr}(R) = \mu\left[a\omega^{-2}-b\omega^2+2\alpha\omega^{-2}(\omega^{-2}-1)+3\beta\omega^{-2}(\omega^{-2}-1)^2\right]-p(R), \tag{20a}$$

$$\tau_{\theta\theta}(R) = \tau_{\theta\theta}(R) = \mu(a\omega-b\omega^{-1})-p(R). \tag{20b}$$

On substituting (20a,b) in (10) and integrating the obtained equation from 0 to R, we have

$$\tau_{rr}(R)-\tau_{rr}(0+) = -2\mu K(c,R), \tag{21}$$

where

$$K(c,R) = \int_0^R \frac{\left[a(\omega^{-2}-\omega)+b(\omega^{-1}-\omega^2)+2\alpha\omega^{-2}(\omega^{-2}-1)+3\beta\omega^{-2}(\omega^{-2}-1)^2\right]}{\omega^3}\frac{\mathrm{d}\xi}{\xi}.$$

In the above integration, ω denotes $\omega(\xi,c) = (1+c^3/\xi^3)^{1/3}$. On substituting (20a) in (21), we have

$$p(R) = \mu\left[a\omega^{-2}-b\omega^2+2\alpha\omega^{-2}(\omega^{-2}-1)+3\beta\omega^{-2}(\omega^{-2}-1)^2\right]+2\mu K(c,R)-\tau_{rr}(0+), \tag{22}$$

where $r(0+) = c$ and $\tau_{rr}(0+)$ are determined from the boundary conditions (6) and (7), moreover,

$$\tau_{rr}(0+) = p_0\left(\frac{A}{\left(A^3+c^3\right)^{1/3}}\right)^2+2\mu K(c,A), \tag{23}$$

$$cp_0 = -2\mu c\left(1+\frac{c^3}{A^3}\right)^{2/3}K(c,A). \tag{24}$$

For the prescribed dead load $p_0 > 0$, if $c \geq 0$ is a solution of (24), then $\tau_{rr}(0+)$ can be obtained by (23). For any prescribed dead load $p_0 > 0$, $c = 0$ is a solution of (24) and $\tau_{rr}(0+)$ is given by (23), so $r(R) = R$ and $p(R) = -p_0$ are the trivial solutions of the spherical symmetric deformation for an incompressible hyper-elastic sphere. If $c > 0$ is a solution of (24), we have $\tau_{rr}(0+) = 0$ from (24), then (4) and (22) are the non trivial solutions of the problem.

We now discuss the existent conditions and the qualitative properties of the nonzero solutions for (24). In what follows, it is convenient to introduce the following dimensionless quantities:

$$\frac{p_0}{\mu} = P, \quad \frac{c}{A} = x. \tag{25}$$

For $K(c, A)$ in (24), replacing ξ in $K(c, A)$ by $\omega = \omega(\xi, c) = (1 + c^3/\xi^3)^{1/3}$, we have

$$K(c, A) = \int_{(1+c^3/A^3)^{1/3}}^{\infty} \frac{\left[a(\omega^{-3}-1)+b(\omega^{-2}-\omega)+2\alpha\omega^{-3}(\omega^{-2}-1)+3\beta\omega^{-3}(\omega^{-2}-1))^2\right]}{\omega^3 - 1} \mathrm{d}\omega. \tag{26}$$

It is not difficult to rewrite (24) as

$$L(P, x, \alpha, \beta) = x\,[l_1(x) + \alpha l_2(x) + \beta l_3(x) - P] = 0, \tag{27}$$

where

$$l_1(x) = 2(1 + x^3)^{2/3}\left[\frac{1}{2}a(1 + x^3)^{-2/3} + b(1 + x^3)^{-1/3}\right], \tag{28a}$$

$$\begin{aligned} l_2(x) = \ & 4(1 + x^3)^{2/3}\left\{-\frac{5\pi}{6\sqrt{3}} + (1 + x^3)^{-1/3} - \frac{1}{2}(1 + x^3)^{-2/3} + \frac{1}{4}(1 + x^3)^{-4/3} + \right. \\ & \left. \frac{1}{\sqrt{3}}\left[\arctan\left(\frac{1 + 2(1 + x^3)^{1/3}}{\sqrt{3}}\right) + \arctan\left(\frac{\sqrt{3}(1 + x^3)^{1/3}}{2 + (1 + x^3)^{1/3}}\right)\right]\right\}, \end{aligned} \tag{28b}$$

$$\begin{aligned} l_3(x) = \ & 6(1 + x^3)^{2/3}\left\{\frac{4\pi}{3\sqrt{3}} - 2(1 + x^3)^{-1/3} + \frac{1}{2}(1 + x^3)^{-2/3} + \frac{1}{3}(1 + x^3)^{-1} \right. \\ & -\frac{1}{2}(1 + x^3)^{-4/3} + \frac{1}{6}(1 + x^3)^{-2} - \frac{1}{\sqrt{3}}\left[2\arctan\left(\frac{1 + 2(1 + x^3)^{1/3}}{\sqrt{3}}\right) + \right. \\ & \left.\left. \arctan\left(\frac{\sqrt{3}(1 + x^3)^{1/3}}{2 + (1 + x^3)^{1/3}}\right)\right] + \frac{1}{2}\ln\frac{(1 + x^3)^{2/3} + (1 + x^3)^{1/3} + 1}{(1 + x^3)^{2/3}}\right\}. \end{aligned} \tag{28c}$$

Obviously, all $l_i(x)$ $(i = 1, 2, 3)$ are sufficient smooth functions on $[0, \infty)$.

Equation (27) gives an exact relation between the dimensionless surface dead load P and the cavity radius x. We call (27) *the cavitated bifurcation equation.*

3.1. QUALITATIVE PROPERTIES OF CAVITATED BIFURCATION EQUATION (27)

3.1.1. *Main results*

It is easy to see that $x \equiv 0$ is a solution of (27) for arbitrarily prescribed $P > 0$. This solution corresponds to the homogeneous state of deformation of the sphere, namely, $r(R) = R$ and $p(R) = -p_0$. So we call $x \equiv 0$ the trivial solution of (27). In order to prove that there is a unique bifurcation point on the trivial solution branch, we consider Equation $L_x(0, P, \alpha, \beta) = 0$. Obviously, it has a unique solution

$$\begin{aligned} P_{\text{cr}} &= l_1(0) + \alpha l_2(0) + \beta l_3(0) = a + 2b + \left(3 - \frac{4\sqrt{3}\pi}{9}\right)\alpha - (9 - \sqrt{3}\pi - 3\log\,3)\beta) \\ &\approx a + 2b + 0{\cdot}5816\alpha - 0{\cdot}2628\beta \end{aligned} \tag{29}$$

From $L_{xP}(0, P_{\text{cr}}, \alpha, \beta) = -1$, we have that $(0, P_{\text{cr}}, \alpha, \beta)$ is the unique bifurcation point of the trivial solution. Furthermore, it is easy to obtain

$$L_{xx}(0, P_{\mathrm{cr}}, \alpha, \beta) = L_{xxx}(0, P_{\mathrm{cr}}, \alpha, \beta) = 0, \tag{30a}$$

$$\begin{aligned} L_{xxxx}(0, P_{\mathrm{cr}}, \alpha, \beta) &= 4\left[l_1''''(0) + \alpha l_2''''(0) + \beta l_3''''(0)\right] \\ &= 24\left[\frac{2}{3}b + \frac{2(15-4\sqrt{3}\pi)}{27}\alpha - \left(6 - \frac{2\sqrt{3}\pi}{3} - 2\log\,3\right)\beta\right] \\ &\approx 24(0{\cdot}6667b - 0{\cdot}5012\alpha - 0{\cdot}1752\beta). \end{aligned} \tag{30b}$$

Since we consider mainly the effect of the material parameters (*i.e.*, α and β) on cavity formation and growth for spheres composed of transversely isotropic hyper-elastic materials (16), here we assume that a and b are two prescribed constants.

From the above analyses, we obtain the following results:

Conclusion 1 For any prescribed $P > 0$, there is a unique bifurcation point on the trivial solution $x \equiv 0$, which corresponds to the critical dead load P_{cr} given by (29) as a cavity occurs in the interior of the sphere. It is clear that the critical load of sphere is given by $P_{\mathrm{cr}} = a + 2b$ when $\alpha = \beta = 0$. It is not difficult to see that, when $0{\cdot}5816\alpha - 0{\cdot}2628\beta > 0$ (or < 0), the cavitated bifurcation for the sphere composed of this class of materials occurs later (or earlier) than that for the isotropic hyper-elastic material.

Conclusion 2 When $L_{xxxx} = (0, P_{\mathrm{cr}}, \alpha, \beta) > 0$ (or < 0), the nontrivial solution of (27) bifurcates locally to the right (or to the left) from the trivial solution at the bifurcation point $(0, P_{\mathrm{cr}}, \alpha, \beta)$. In particular, as $L_{xxxx} = (0, P_{\mathrm{cr}}, \alpha, \beta) < 0$, there exists a secondary turning bifurcation point on the nontrivial solution branch of (27) that bifurcates locally to the left.

In fact, from (30a,b), we see that $L_x(x, P, \alpha, \beta) < 0$ for sufficient small $x > 0$. It is not difficult to show that $L_x(x, P, \alpha, \beta) > 0$ for sufficient large x. Thus, we may conclude that there must exist a value x_n such that $L_x(x_n, P, \alpha, \beta) = 0$, and the corresponding dead load is written as P_n, that is to say, there is a secondary turning bifurcation point (x_n, P_n) on the nontrivial solution branch.

The foregoing results generalize those of Polignone and Horgan [7] who considered cavity formation and growth for spheres composed of a one-parameter family of transversely isotropic materials.

3.1.2. *Equivalent normal forms*

From (27), we know that $L(0, P_{\mathrm{cr}}, \alpha, \beta) = 0$; furthermore, we have $L_p(0, P_{\mathrm{cr}}, \alpha, \beta) = 0$, $L_{xP}(0, P_{\mathrm{cr}}, \alpha, \beta) = -1$ as well as (30a,b). In terms of the distinguished conditions for the bifurcation equation in the singularity theory (*cf.* [34, Chapter 3]) and $P > 0$, $x \geq 0$, we obtain the following conclusion.

Conclusion 3 If $l_1''''(0) + \alpha l_2''''(0) + \beta l_3''''(0) \neq 0$, then $L(x, P, \alpha, \beta)$ is equivalent to the normal forms $\pm x^4 - \delta x$ with single-sided constraint conditions at the critical point $(0, P_{\mathrm{cr}}, \alpha, \beta)$.

To study the effect of the material parameters α, β on the solutions of the cavitated bifurcation equation, we divide the parameter plane α, β into four regions (See Figure 1) by the following two straight lines

$$k_1(\alpha, \beta) = \left(3 - \frac{4\sqrt{3}\pi}{9}\right)\alpha - (9 - \sqrt{3}\pi - 3\log\,3)\beta = 0 \tag{31a}$$

and

$$k_2(\alpha, \beta) = \frac{2}{3}b + \frac{2(15-4\sqrt{3}\pi)}{27}\alpha - \left(6 - \frac{2\sqrt{3}\pi}{3} - 2\log\,3\right)\beta = 0, \tag{31b}$$

where (31a) means that the critical dead load corresponding to the material (16) is smaller or larger than that of the isotropic material, while (31b) determines the bifurcation direction of the nontrivial solution of the cavitated bifurcation equation at the critical point. For convenience and certainty, in the following calculation, we take $a = 1$, $b = 1$. The four regions in the parameter plane α, β are denoted by

$$\Omega_1 = \{(\alpha, \beta) : k_1(\alpha, \beta) > 0, \quad k_2(\alpha, \beta) > 0, \quad \beta \geq 0\}\,; \tag{32a}$$

$$\Omega_2 = \{(\alpha, \beta) : k_1(\alpha, \beta) < 0, \quad k_2(\alpha, \beta) > 0, \quad \alpha \geq 0\}\,; \tag{32b}$$

$$\Omega_3 = \{(\alpha, \beta) : k_1(\alpha, \beta) < 0, \quad k_2(\alpha, \beta) < 0, \quad \alpha \geq 0\}\,; \tag{32c}$$

$$\Omega_4 = \{(\alpha, \beta) : k_1(\alpha, \beta) > 0, \quad k_2(\alpha, \beta) < 0, \quad \beta \geq 0\}\,. \tag{32d}$$

Thus, we have the following conclusion.

Conclusion 4 When the parameters (α, β) belong to Ω_2 or Ω_3 (Ω_1 or Ω_4), the critical load associated with the strain-energy function (16) is smaller (larger) than that associated with $\alpha = \beta = 0$ in (16); when the parameters (α, β) belong to Ω_3 or Ω_4 (Ω_1 or Ω_2), the nontrivial solution of the cavitated bifurcation equation bifurcates locally to the left (right) at the critical point, and there exists a secondary turning bifurcation point on the nontrivial solution when the parameters belong to Ω_3 or Ω_4.

Remark. From (31b), we see that, in Figure 1, when the value of b increases from 0 to 2, the regions Ω_1 and Ω_2 increase from the origin to the maximum regions, and the boundary-line as $b = 2$ is the dot line shown in Figure 1. In fact, there is also a line, $a + 2b + 0{\cdot}5816\alpha - 0{\cdot}2628\beta = 0$, in Figure 1. Since we only consider small parameters (α, β) and this line is far from the regions we partitioned, this line is not considered.

3.1.3. *Numerical examples*

Next we show the cavity formulation and growth in the interior of the sphere by several numerical examples when the material parameters (α, β) be long to different regions.

(1)As $\alpha = 1$, $\beta = 0{\cdot}5$ in region Ω_1, we obtain the critical load is $P_{\rm cr} = 3{\cdot}4502$.

(2) As $\alpha = 0{\cdot}5$, $\beta = 1{\cdot}5$ in region Ω_2, we obtain $P_{\rm cr} = 2{\cdot}8966$.

(3) As $\alpha = 1$, $\beta = 3$ in region Ω_3, we obtain $P_{\rm cr} = 2{\cdot}7932$ and the coordinates of the secondary turning bifurcation point is $(x_n, P_n) = (2{\cdot}7696, 0{\cdot}5248)$.

(4) As $\alpha = 2$, $\beta = 1$ in region Ω_4, we obtain $P_{\rm cr} = 3{\cdot}9004$ and $(x_n, P_n) = (3{\cdot}8191, 0{\cdot}7445)$.

Curves of the relation between the dimensionless surface load and the cavity radius in the four regions are shown in Figures 2–5.

3.2. STABILITY AND CATASTROPHE OF SOLUTIONS

To examine the stability of the solutions of Equation (27), we now carry out an energy analysis. For the incompressible hyper-elastic material (16), the total energy corresponding to any equilibrium configuration of the sphere is given by

$$\begin{aligned} E(C) &= 4\pi \int_0^A W R^2 \mathrm{d}R - 4\pi A^2 p_0 [r(A) - A] \\ &= 4\pi c^3 \int_{(1+c^3/A^3)^{1/3}}^{\infty} \frac{\omega^2 \Sigma(\omega)}{(\omega^3 - 1)^2} \mathrm{d}\omega - 4\pi A^2 p_0 \left[(A^3 + c^3)^{1/3} - A\right], \end{aligned} \tag{33}$$

where $\Sigma(\omega)$ is given by (19). The first term in (33) is the total strain energy, and the second term is the work done by the tensile dead load. The dimensionless form of (33) is

$$\Lambda(x) = \frac{E(x)}{(4/3)\pi A^3\mu} = \psi(x) + P\gamma(x), \tag{34}$$

where

$$\psi(x) = \frac{3x^3}{\mu}\int_{(1+x^3)^{1/3}}^{\infty} \frac{\omega^2\Sigma(\omega)}{(\omega^3-1)^2}\mathrm{d}\omega, \quad \gamma(x) = -3\left[(1+x^3)^{1/3}-1\right].$$

For the trivial solution $x = 0$, from (34), it is not difficult to obtain

$$\Lambda(0) = \Lambda'(0) = \Lambda''(0) = 0, \quad \Lambda'''(0) = 6(P_{\mathrm{cr}} - P), \tag{35}$$

where Λ' denotes the derivative with respect to x. The trivial solution minimizes $\Lambda(0)$ among all possible (spherical symmetric) configurations as $P < P_{\mathrm{cr}}$ thus it is stable. But for $P > P_{\mathrm{cr}}$ the trivial solution provides a local maximum for $\Lambda(0)$ among all possible (spherical symmetric) configurations, thus it is unstable.

For the nontrivial solution, we have a Taylor expansion of (34) at $x = 0as$ follows:

$$\Lambda(x) = -\frac{1}{2}(0{\cdot}6667 - 0{\cdot}5012\alpha - 0{\cdot}1752\beta)x^6 + O(x^9). \tag{36}$$

From Conclusion 1, we know that, for $P > P_{\mathrm{cr}}$, the nontrivial solution of the cavitated bifurcation equation (27) will bifurcate at the critical point $x = 0$, $P = P_{\mathrm{c}}$ from the trivial solution $x \equiv 0$. Equation (36) shows the following: if the nontrivial solution of (27) bifurcates to the left (namely, the parameters (α, β) belong to Ω_3 or Ω_4) near $x = 0$, $P = P_{\mathrm{cr}}$, the total energy takes the local maximum, thus the nontrivial solution is unstable. Similarly, if the nontrivial solution of (27) bifurcates to the right (namely, the parameters (α, β) belong to Ω_1 or Ω_2) near $x = 0$, $P = P_{\mathrm{cr}}$, the total energy assumes a local minimum, and the nontrivial solution here is stable. Curves for the relation between the energy and the cavity radius in Ω_2 and Ω_4 are shown in Figures 6 and 7, respectively. The energy curves in Ω_1 and Ω_3 are similar to those of Ω_2 and Ω_4, respectively.

In summary, when the parameters (α, β) belong to Ω_1 or Ω_2, the trivial solution of (27) is stable for $P < P_{\mathrm{cr}}$ and it is unstable for $P > P_{\mathrm{cr}}$. After a cavity forms, the nontrivial solution, which increases monotonically, is stable. When the parameters (α, β) belong to Ω_3 or Ω_4, the stability of the solutions changes since there is a secondary turning point on the nontrivial solution branch, as shown in Figure 8. The number of solutions, corresponding to the number of equilibrium states, here depends on the values of P in the following way: when $0 < P < P_n$, there is only one stable trivial solution $x \equiv 0$; when $P_n < P < P_{\mathrm{cr}}$, there are exactly three solutions, among which one stable trivial solution $x \equiv 0$ and two nontrivial solutions x_1 and x_2 with $0 < x_1 < x_n < x_2$. It is easy to show that x_1 maximizes the total energy and hence it is unstable, x_2 minimizes the total energy and hence it is stable; when $P > P_{\mathrm{cr}}$, there are exactly two solutions, namely, one unstable trivial solution $x \equiv 0$ and one stable nontrivial solution $x > x_c$. However, there are two stable equilibrium solutions as $P_n < P < P_{\mathrm{cr}}$; they correspond to the trivial solution $x \equiv 0$ and the nontrivial solution x_2, respectively. To illustrate which solution does correspond to the actual stable equilibrium state, we have to solve; equation $\Lambda(x) = 0$. It is not difficult to obtain a positive value x_t corresponding to P_t; see Figure 7. Further, as $P_n < P < P_t$, the trivial solution provides a local minimum for the total energy, and thus it is the actual stable equilibrium state; as

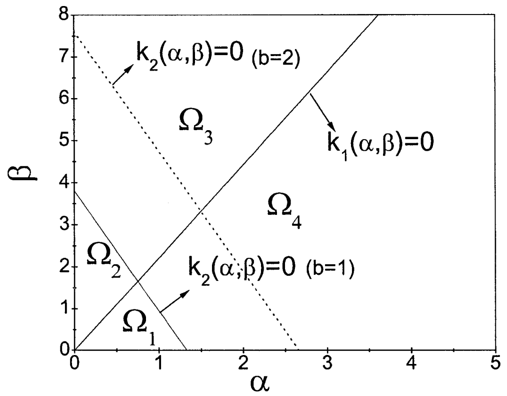

Figure 1. Regions partitioned in parameter plane.

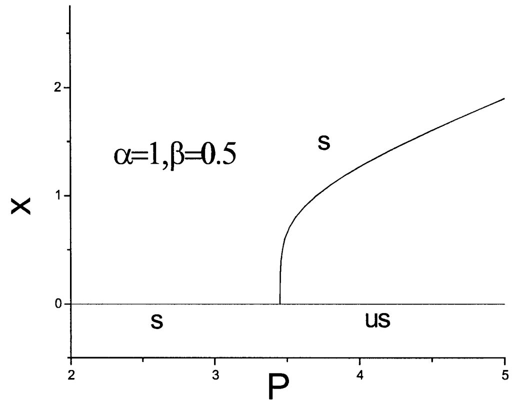

Figure 2. Solution curves of Equation (27) and their stabilities in Ω_1.

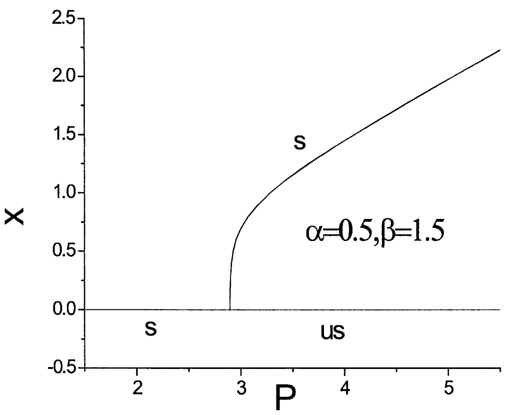

Figure 3. Solution curves of Equation (27) and their stabilities in Ω_2.

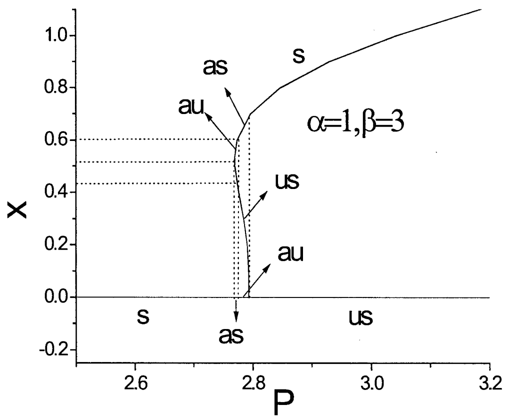

Figure 4. Solution curves of Equation (27) and their stabilities in Ω_3.

$P_t < P < P_{\mathrm{cr}}$, the nontrivial solution here provides a local minimum for the total energy, and it is the actual stable equilibrium state.

In Ω_3 or Ω_4, the solutions of the cavitated bifurcation equation (27) can also give rise to catastrophe phenomena. As shown in Figure 8, if the surface tensile dead load P changes quasi-statically from small to bigger, then for $P < P_t$, no cavity occurs in the interior of the sphere; but when $P > P_t$, the equilibrium state of the sphere changes suddenly, namely, a bigger cavity occurs. In contrast, if the dead load P changes quasi-statically from big to small, the cavity radius reduces suddenly to zero only as $P < P_t$.

Note: In Figures 2–5, ‘s’, ‘u’, ‘as’, ‘au’ denote ‘stable’, ‘unstable’, ‘actual stable’, ‘actual unstable’, respectively.

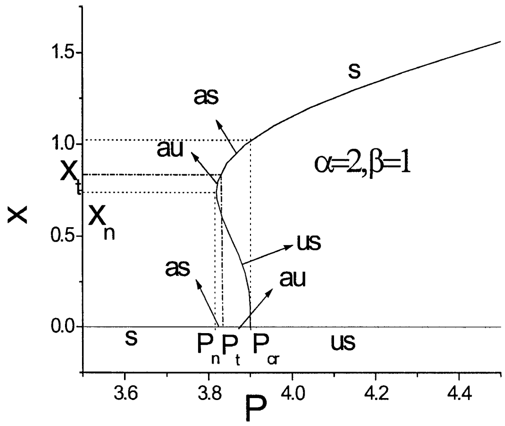

Figure 5. Solution curves of Equation (27) and their stabilities in Ω_4.

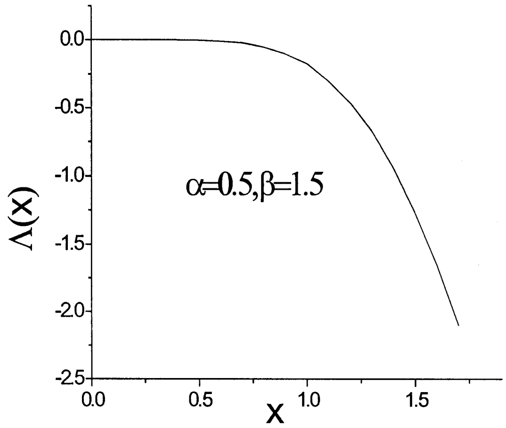

Figure 6. Energy curve in Ω_1 and Ω_2.

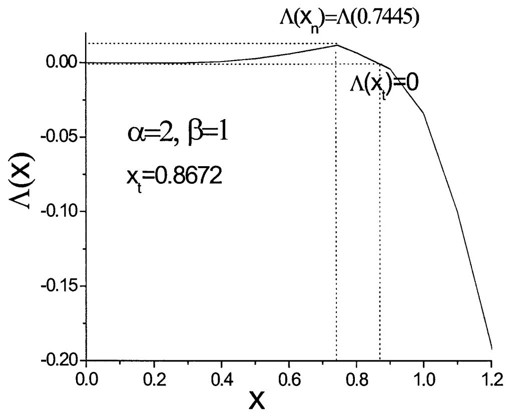

Figure 7. Energy curve in Ω_3 and Ω_4.

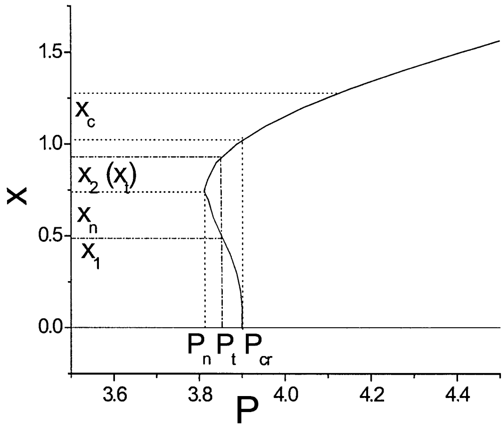

Figure 8. Sketch map for stability and catastrophe of solutions in Ω_3 and Ω_4.

3.3. Stress concentration and catastrophe

On substituting (22) and (26) in (20a, b), we have

$$\begin{aligned}\hat{\tau}_{rr}(\rho) &= P\left(1+x^3\right)^{-2/3} + 2K(x) - 2K(x,\rho),\\ \hat{\tau}_{\theta\theta}(\rho) &= \hat{\tau}_{\phi\phi}(\rho) = \omega(\rho,x) - \omega^{-1}(\rho,x) - \omega^{-2}(\rho,x) + \omega^2(\rho,x)\\ &\quad -2\alpha\omega^{-2}(\rho,x)(\omega^{-2}(\rho,x)-1) - 3\beta\omega^{-2}(\rho,x)(\omega^{-2}(\rho,x)-1)^2 + \hat{\tau}_{rr}(\rho),\end{aligned}$$

where $\hat{\tau}_{rr}(\rho) = \tau_{rr}(\rho)/\mu$, $\hat{\tau}_{\theta\theta}(\rho)/\mu$, $\rho = R/A$.

In Ω_1 or Ω_2, from the above analyses, one has that, for $P \leq P_{\text{cr}}$, no cavity forms in the interior of the sphere; Equation (27) has only the trivial solution, namely, $x \equiv 0$, so we have

$$\hat{\tau}_{rr}(\rho) = \hat{\tau}_{\theta\theta}(\rho) = \hat{\tau}_{\phi\phi}(\rho) = P.$$

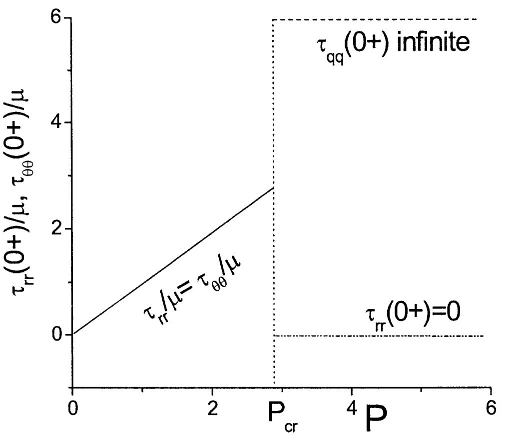

Figure 9. Stress jumping at cavity surface.

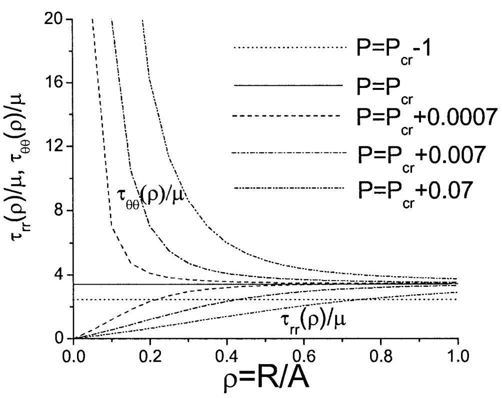

Figure 10. Stress distribution curves for $\alpha = 1$, $\beta = 0{\cdot}5$.

However, for the prescribed $P > P_{\rm cr}$, there is a unique stable nontrivial solution of (27), that is to say, a cavity forms at the center of the sphere. From (24), we obtain

$$\begin{aligned}
\hat{\tau}_{rr}(\rho) &= -2K(x,\rho),\\
\hat{\tau}_{\theta\theta}(\rho) &= \hat{\tau}_{\phi\phi}(\rho) = \omega(\rho,x) - \omega^{-1}(\rho,x) - \omega^{-2}(\rho,x) + \omega^{2}(\rho,x)\\
&\quad -2\alpha\omega^{-2}(\rho,x)(\omega^{-2}(\rho,x)-1) - 3\beta\omega^{-2}(\rho,x)(\omega^{-2}(\rho,x)-1)^2 - 2K(x,\rho).
\end{aligned}$$

If a cavity forms in the interior of the sphere, the stresses near the cavity $R = 0+$ are given by $\hat{\tau}_{rr}(0+) = 0$ and $\hat{\tau}_{\theta\theta}(0+) = \hat{\tau}_{\phi\phi}(0+) = +\infty$, respectively.

The discontinuity of the stresses, $\hat{\tau}_{rr}(0+)$ and $\hat{\tau}_{\theta\theta}(0+)$, for the prescribed P is shown in Figure 9. For different prescribed values of P, the distributions of the radial and circumferential stresses are shown in Figure 10. From Figure 10, one can see that, for $P < P_{\rm cr}$, both the radial stress $\hat{\tau}_{rr}(\rho)$ and the circumferential stress $\hat{\tau}_{\theta\theta}(\rho)$ are homogeneous and increasing with P. However, as $P = P_{\rm cr}$, the stresses undergo an obvious catastrophic transition from the homogeneous distribution to the non-homogeneous distribution. If $P > P_{\rm cr}$, the value of the circumferential stress near the cavity is infinite, but it decreases rapidly as ρ increases gradually, and there obviously exists a concentration phenomenon for the circumferential stress near the cavity. This implies that the stress concentration is a local phenomenon; however, this is just the reason for the sudden appearance of the cavity and its subsequent rapid growth, as one would expect on physical grounds.

For Ω_3 or Ω_4, the discussion on the radial and circumferential stresses are similar. Further, Polignone and Horgan [7, 12, 13], Ren and Cheng [8, 9] have also discussed this class of phenomena.

4. Cavitated bifurcation for compressible hyper-elastic sphere

4.1. Solutions and their qualitative properties

On substituting (2) and (8) in (10), the equilibrium equation for the radial deformation function $r(R)$ is denoted by

$$\frac{\partial^2 W}{\partial \lambda_1^2}\ddot{r}(R) + \frac{2}{R}\left[\frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2}\left(\dot{r}(R) - \frac{r(R)}{R}\right) + \left(\frac{\partial W}{\partial \lambda_1} - \frac{\partial W}{\partial \lambda_2}\right)\right] = 0, \tag{37}$$

where the strain-energy function W is given by (17).

Obviously, for arbitrary prescribed $\lambda > 1$, one solution of (37), called the 'homogeneous solution' satisfying the boundary conditions (6) and (9), is denoted by

$$r(R) = \lambda R, \tag{38}$$

which corresponds to the homogeneous radial displacement $u(R) = (\lambda - 1)R$. Thus (38) is called the trivial solution.

In order to obtain nontrivial solutions of the problem, let

$$\varepsilon = \varepsilon(R) = \frac{\lambda_1}{\lambda_2} = \frac{R}{r(R)}\dot{r}(R), \quad 0 < R \leq A. \tag{39}$$

It is easy to see that, if $r(0+) = 0$, we have $\varepsilon(0+) = \lim_{R\to 0+} \frac{R}{r(R)}\dot{r}(R) = 1$, while if $r(0+) > 0$, then $\varepsilon(0+) = \lim_{R\to 0+} \frac{R}{r(R)}\dot{r}(R) = 0$. On substituting (39) and the strain-energy function (17) in (37), we obtain the equivalent equations

$$\dot{r}(R) = \frac{r(R)}{R}\varepsilon(R), \qquad \dot{\varepsilon}(R) = \frac{1}{R}\frac{\varepsilon(1-\varepsilon)(1+2\varepsilon)(3+2\varepsilon)}{1+4\varepsilon}. \tag{40-41}$$

From $\varepsilon(R) > 0$, we see that the solutions of Equation (41) only have the following three cases: (1) $\varepsilon(R) \equiv 1$; (2) $\varepsilon(R) > 1$; (3) $0 < \varepsilon(R) < 1$. We now discuss these in detail.

(1) If $\varepsilon(R) \equiv 1$ $(R \in (0, A))$, then we obtain $r(R) = cR$ from (40), where c is an arbitrary positive constant. It is clear that it corresponds to the homogeneous solution (38) from the condition (9).

(2) If $\varepsilon(R) > 1$ $(R \in (0, A))$, then (41) shows that $\dot{\varepsilon}(R) < 0$, namely, ε decreases monotonously with R. If $r(0+) > 0$, we have $\varepsilon(0+) = 0$ from the above analysis; thus we have $\varepsilon(R) < 0$. However, this leads to $\dot{r}(R) < 0$, which contradicts with the condition (3). If $r(0+) = 0$, here $r(R) = \lambda R$, we have $\varepsilon(R) \equiv 1$; this contradicts with our hypothesis. Thus this case is impossible.

(3) If $0 < \varepsilon(R) < 1$ $(R \in (0, A))$, then (41) shows that $\dot{\varepsilon}(R) > 0$, namely ε increases monotonously with R. Thus we have $\varepsilon(0+) = 0$. Let $\varepsilon(A) = \varepsilon_0$, here ε_0 is an undetermined constant satisfying the condition $0 < \varepsilon_0 < 1$. The inverse function of $\varepsilon(R)$ is denoted by $R = R(\varepsilon)$, $(\varepsilon \in (0, \varepsilon_0])$. It is obvious that $R(\varepsilon_0) = A$. Integration of Equation (41) from ε to ε_0 yields

$$R(\varepsilon, \varepsilon_0) = A\left(\frac{\varepsilon}{\varepsilon_0}\right)^{1/3}\left(\frac{1-\varepsilon_0}{1-\varepsilon}\right)^{1/3}\left(\frac{1+2\varepsilon}{1+2\varepsilon_0}\right)^{1/3}\left(\frac{3+2\varepsilon_0}{3+2\varepsilon}\right)^{1/3}. \tag{42}$$

Let $\hat{r}(\varepsilon, \varepsilon_0, \lambda) = r(R(\varepsilon, \varepsilon_0), \lambda)$. On substituting (42) in (40) and integrating the obtained formula from ε to ε_0, we have

$$\hat{r}(\varepsilon, \varepsilon_0, \lambda) = \lambda A \left(\frac{1-\varepsilon_0}{1-\varepsilon}\right)^{1/3} \left(\frac{1+2\varepsilon}{1+2\varepsilon_0}\right)^{1/6} \left(\frac{3+2\varepsilon_0}{3+2\varepsilon}\right)^{1/2}. \tag{43}$$

Further, it is easy to obtain that the principal stretches and the principal stresses are as follows:

$$\lambda_2(\varepsilon, \varepsilon_0, \lambda) = \frac{R(\varepsilon, \varepsilon_0)}{\hat{r}(\varepsilon, \varepsilon_0, \lambda)} = \lambda \left(\frac{\varepsilon_0}{\varepsilon}\right)^{1/3} \left(\frac{1+2\varepsilon_0}{1+2\varepsilon}\right)^{1/2} \left(\frac{3+2\varepsilon}{3+2\varepsilon_0}\right)^{5/6}, \tag{44}$$

$$\lambda_1(\varepsilon, \varepsilon_1, 0, \lambda) = \varepsilon\lambda_2(\varepsilon, \varepsilon_0, \lambda). \tag{45}$$

$$\tau_{rr}(\varepsilon, \varepsilon_0, \lambda) = \left[\frac{c_1}{\lambda^2}\left(\frac{\varepsilon}{\varepsilon_0}\right)^{2/3}\left(\frac{1+2\varepsilon}{1+2\varepsilon_0}\right)\left(\frac{3+2\varepsilon_0}{3+2\varepsilon}\right)^{5/3} - \frac{c_2}{\lambda^3\varepsilon_0}\frac{(1+2\varepsilon)^{1/2}}{(1+2\varepsilon_0)^{3/2}}\left(\frac{3+2\varepsilon_0}{3+2\varepsilon}\right)^{5/2} + c_3\right], \tag{46}$$

$$\tau_{\theta\theta}(\varepsilon, \varepsilon_0, \lambda) = \left[\frac{c_1}{\lambda^2\varepsilon}\left(\frac{\varepsilon}{\varepsilon_0}\right)^{2/3}\left(\frac{1+2\varepsilon}{1+2\varepsilon_0}\right)\left(\frac{3+2\varepsilon_0}{3+2\varepsilon}\right)^{5/3} - \frac{c_2\varepsilon}{\lambda^3\varepsilon_0}\frac{(1+2\varepsilon)^{1/2}}{(1+2\varepsilon_0)^{3/2}}\left(\frac{3+2\varepsilon_0}{3+2\varepsilon}\right)^{5/2} + c_3\right]. \tag{47}$$

Equations (43–47) give the parameter-type solutions for the radial deformation, the principal stretches and the principal stresses. From condition (6) at the center of the sphere, namely, $\varepsilon = 0$, we have

$$\eta\tau_{rr}(0, \varepsilon_0, \lambda) = 0, \tag{48}$$

here

$$\eta = \hat{r}(0, \varepsilon_0, \lambda) = \lambda A(1-\varepsilon_0)^{1/3}(1+2\varepsilon_0)^{1/6}\left(\frac{3}{3+2\varepsilon_0}\right)^{1/2}, \tag{49}$$

$$\tau_{rr}(0, \varepsilon_0, \lambda) = \left[-\frac{c_2}{\lambda^3\varepsilon_0}\frac{1}{(1+2\varepsilon_0)^{3/2}}\left(\frac{3+2\varepsilon_0}{3}\right)^{5/2} + c_3\right]. \tag{50}$$

The formulation (48) gives a relation between the stretch λ and the cavity radius η. So we call (48) *the cavitated bifurcation equation*. From (49) we have that $\varepsilon_0 = 1$ is the unique solution of $\eta = 0$. Thus we have $\eta = 0 \Leftrightarrow 1 - \varepsilon_0 = 0$. Further, (48) is equivalent to $\Phi(\varepsilon_0, \lambda) = (1-\varepsilon_0)\tau_{rr}(\varepsilon_0, \lambda) = 0$. Let's consider $\Phi_{\varepsilon_0}(1, \lambda) = 0$. It is easy to obtain the unique solution of this equation, namely,

$$\lambda_{\mathrm{cr}} = \left(\frac{5^{5/2}c_2}{81c_3}\right)^{1/3} = \left(\frac{5^{5/2}}{9}\frac{1-\nu}{1+4\nu}\right)^{1/3}. \tag{51}$$

It is not difficult to show that $\Phi_{\varepsilon_0\lambda}(1, \lambda_{\mathrm{cr}}) \neq 0$, thus we have that $(R, \lambda) = (0, \lambda_{\mathrm{cr}})$ is a unique bifurcation point on the homogeneous solution $r(R) = \lambda R$.

When a cavity forms at the center of the sphere, *i.e.*, $\lambda > 0$, from $\tau_{rr}(0+) = 0$, we have

$$\lambda^3 = \frac{c^2}{c_3\varepsilon_0}\frac{1}{(1+2\varepsilon_0)^{3/2}}\left(\frac{3+2\varepsilon_0}{3}\right)^{5/2}. \tag{52}$$

In summary, for arbitrarily prescribed $\lambda > 1$, $r(R) = \lambda R$ is a homogeneous solution of (37), and there is a unique bifurcation point $(0, \lambda_{cr})$ on it, where the critical stretch λ_{cr} is given by (51). When $\lambda > \lambda_{cr}$, a cavity forms in the interior of the sphere, (42), (43) and (52) are the parameter-type cavitated bifurcation solutions of Equation (37) satisfying (6) and (9) associated with the strain-energy function (17). For $\varepsilon_0 = 1$, $r(0+) = 0$ is a solution of Equation (48). When $0 < \varepsilon_0 < 1$, for any arbitrary prescribed $\lambda > \lambda_{cr}$, we can obtain a value of ε_0 from (52). Substituting this in (49), we can obtain a value of cavity radius. From (51), one can see that the critical stretch λ_{cr} decreases monotonically with increasing the Poisson ratio ν, where $0 < \nu < 2/7$. For different Poison ratios ν, curves for the cavity radius and the stretch are shown in Figure 11. In contrast to the situation described in [19], the critical stretch for the cavity formation for the strain energy function (17) in our paper is smaller than that in [19] for the same Poisson ratio. In other words, cavitated bifurcation for the sphere composed of this kind of hyper-elastic materials occurs earlier than for that in [19].

4.2. Stability of Solutions

For $\lambda > \lambda_{cr}$, there are two solutions of Equation (37), namely, a homogeneous solution and a cavitated bifurcation solution. We now examine their relative stabilities from the total potential energy.

The total potential energy of the sphere subjected to a prescribed uniform radial displacement on its surface is given by

$$E(\lambda) = 4\pi\int_0^A R^2 W \mathrm{d}R = \frac{4\pi}{3}A^3\left\{W(\lambda_1,\lambda_2,\lambda_3) - (\lambda_1-\lambda_2)\frac{\partial W}{\partial\lambda_1}(\lambda_1,\lambda_2,\lambda_3)\right\}. \tag{53}$$

For the homogeneous solution, we have $\dot{r}(A) = \lambda$; and for the cavitated bifurcation solution, we have $\dot{r}(A) = \lambda\varepsilon_0$. Thus, the total potential energies of the sphere for the homogeneous solution and the cavitated bifurcation solution are given by

$$E_h(\lambda) = \frac{4\pi A^3}{3}W(\lambda,\lambda,\lambda) \tag{54}$$

and

$$E_c(\lambda) = \frac{4\pi A^3}{3}\left\{W(\lambda\varepsilon_0,\lambda,\lambda) - \lambda(\varepsilon_0-1)\frac{\partial W}{\partial\lambda_1}(\lambda\varepsilon_0,\lambda,\lambda)\right\}, \tag{55}$$

respectively.

Substituting (17) in (53) and (54) yields

$$E_c(\lambda) - E_h(\lambda) = \frac{4\pi A^3 c_2}{3}\left[\log\frac{1+2\varepsilon_0}{3\varepsilon_0} - \frac{1-\varepsilon_0}{\varepsilon_0(1+2\varepsilon_0)}\right]. \tag{56}$$

Since $0 < \varepsilon_0 < 1$, it is easy to show that the right-hand side of (56) is negative, since $E_c < E_h$.

Consequently, the energy of the sphere for the cavitated bifurcation solution is strictly less than that of the homogeneous solution for the same λ. Thus, the cavitated bifurcation solution is stable.

According to the formula of radial displacement $u(R,\lambda) = r(R,\lambda) - R$, for different λ, the curves for the radial displacement are shown in Figure 12, where r and R are given by (42)

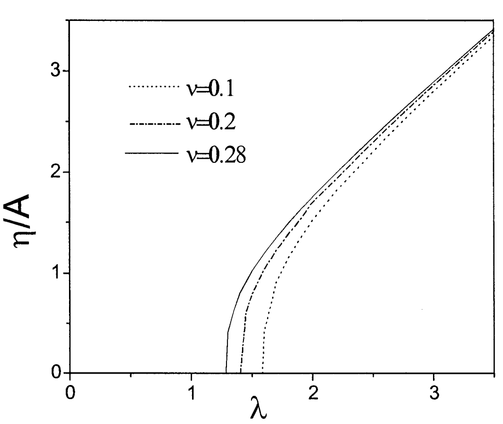

Figure 11. Bifurcation curves of cavity radius for different ν.

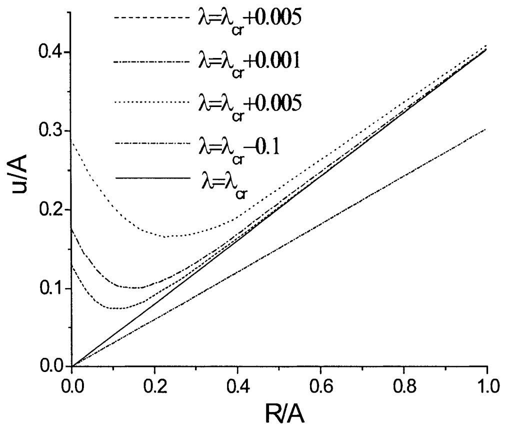

Figure 12. Radial displacement curves for different λ.

and (43), respectively. From Figure 12 we conclude: when $\lambda < \lambda_{\rm cr}$, the deformation state of the sphere is homogeneous, and no jumping of the radial displacement occurs until the critical state $\lambda = \lambda_{\rm cr}$; but when $\lambda > \lambda_{\rm cr}$, the radial displacement may jump and the homogeneous state bifurcates continuously into the cavitated deformation state. Furthermore, the transition for the slope of radial displacement is not continuous, as shown in Figure 12, the cavitated displacement $u(R)$ has positive and negative slopes in the different domains of R. The positive slope of $u(R)$ corresponds to extension while the negative slope corresponds to compression. That is to say: when $\lambda < \lambda_{\rm cr}$, namely, before the cavity forms, the deformation of the whole sphere is extension, but when $\lambda > \lambda_{\rm cr}$, namely, after the cavity forms, the deformation near the cavity becomes compressive. This is obviously different from that of the homogeneous deformation.

4.3. Stress concentration and catastrophe

From the above analyses, one can see that $r(R) = \lambda R$ as $\lambda < \lambda_{\rm cr}$ and we have $\tau_{rr}(R) = \tau_{\theta\theta}(R) = \tau_{\phi\phi}(R) = \dfrac{c_1}{\lambda^2} - \dfrac{c_2}{3\lambda^3} + c_3$. And as $\lambda \geq \lambda_{\rm cr}$, we can calculate exactly the stresses from (46) and (47). Since the analyses for the stresses are similar to that of Section 3.3, we only carry out the numerical calculations. The jumping figures of the stresses at the surface of the cavity and the stress distribution curves are shown in Figures 13 and 14, respectively.

One has to point out that, from Figure 13, as the prescribed stretch $\lambda \geq \lambda_{\rm cr}$, the circumferential stress is finite for the material given in [19], but for the material given by (17), the circumferential stress is infinite.

5. Conclusions

In this paper, we have considered cavitated bifurcation problems for spheres composed of a transversely isotropic incompressible hyper-elastic material and a compressible hyper-elastic material, respectively, and have reached the following conclusions.

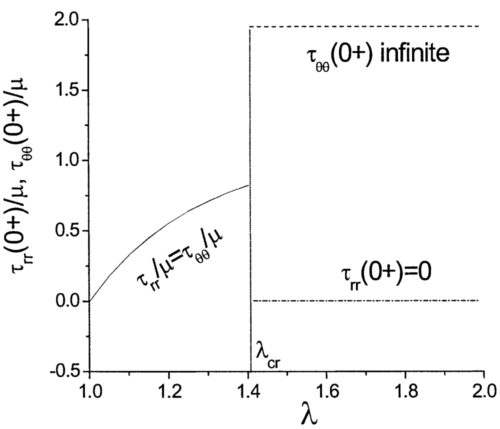

Figure 13. Stress jumping at the cavity surface.

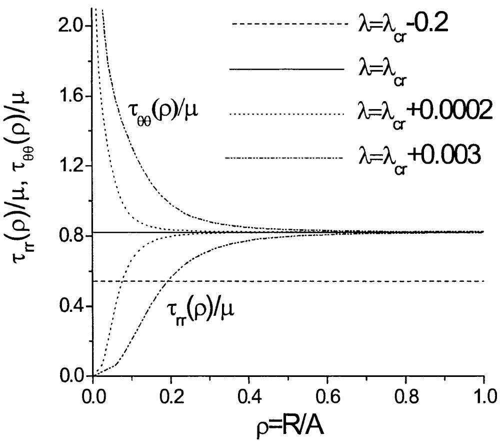

Figure 14. Stress distribution and concentration.

(1) For the transversely isotropic incompressible hyper-elastic solid sphere associated with the strain-energy function (16), we have

(1.1) The cavitated bifurcation can occur for this class of incompressible hyper-elastic spheres as the prescribed uniform dead load exceeds the critical dead load. When the parameters (α, β) belong to Ω_2 or $\Omega_3(\Omega_1$ or $\Omega_4)$, the cavitated bifurcation for the sphere composed of this class of materials occurs earlier (later) than that for the isotropic material (*i.e.*, (16) with $\alpha = \beta = 0$).

(1.2) When the parameters (α, β) belong to Ω_1 or Ω_2 (Ω_3 or Ω_4), the nontrivial solution of the cavitated bifurcation equation bifurcates locally to the right (to the left) of the trivial solution at the bifurcation point $(0, P_{\mathrm{cr}}, \alpha, \beta)$, and there exists a secondary turning bifurcation point on the nontrivial solution branch as the parameters (α, β) belong to Ω_3 or Ω_4.

(1.3) If $l_1''''(0) + \alpha l_2''''(0) + \beta l_3''''(0) \neq 0$, then $L(x, P, \alpha, \beta)$ is equivalent to the normal forms $\pm x^4 - \delta x$ with single-sided constraint conditions at the critical point $(0, P_{\mathrm{cr}}, \alpha, \beta)$.

We point out that, according to singularity theory, if some higher-order radial terms are introduced into the strain-energy function (16), the qualitative properties of the solutions of the cavitated bifurcation equation are also similar.

(2) For the isotropic compressible hyper-elastic sphere associated with the strain-energy function (17), we have the following conclusions:

(2.1) The cavitated bifurcation can occur for this class of compressible hyper-elastic spheres as the prescribed stretch exceeds the critical stretch. A group of parameter-type cavitated bifurcation solutions and the expression of critical stretch are obtained. The critical stretch λ_{cr}, given by (51), decreases monotonously with increasing Poisson ratio ν, where $0 < \nu < 2/7$.

(2.2) The cavitated bifurcation for the hyper-elastic sphere associated with the strain-energy function (17) occurs earlier than that in [19] for the same Poisson ratio. In other words, the cavitated bifurcation for the sphere composed of this kind of hyper-elastic materials occurs earlier than that in [19].

(2.3) When $\lambda < \lambda_{cr}$, namely before the cavity forms, the deformation of the whole sphere is an extension, but when $\lambda > \lambda_{cr}$, namely, after the cavity forms, the deformation near the cavity becomes compressive.

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